

Divisibility Relations for the Dimensions and Hilbert series of Nichols Algebras of Non-Abelian Group Type

Andreas Lochmann

June 29, 2012

Abstract

We present a divisibility relation for the dimensions and Hilbert series of certain classes of Nichols algebras of non-abelian group type, which generalizes Nichols algebras over Coxeter groups with constant cocycle -1 . For this we introduce three groups of isomorphisms acting on Nichols algebras, which generalizes the exchange operator introduced by Milinski and Schneider in [17] for Coxeter groups.

1 Introduction

In [11], Theorem 4.14, Graña, Heckenberger and Vendramin gave a full classification of finite-dimensional Nichols algebras of non-abelian group type with absolutely irreducible Yetter-Drinfeld modules under the assumption that their Hilbert series factorize as

$$\mathcal{H}_{\mathfrak{B}}(t) = \prod_{j=1}^r (\alpha_j)_t \quad \text{with} \quad (\alpha_j)_t := 1 + t + t^2 + \dots + t^{\alpha_j-1}$$

for some $r, \alpha_j \in \mathbb{N}$. From the examples in Subsection 2.3, they covered all Nichols algebras except $\mathfrak{B}(Q_{3,1}, E_3)^{(2)}$ and $\mathfrak{B}(Q_{4,1}, \chi_4)$, whose Hilbert series do not factorize as above. In a subsequent paper ([15]), Heckenberger, Vendramin and the author classified all finite-dimensional Nichols algebras of non-abelian group type with absolutely irreducible Yetter-Drinfeld modules under the more general assumption

$$\mathcal{H}_{\mathfrak{B}}(t) = \prod_{j=1}^r (\alpha_j)_t \cdot \prod_{j=1}^s (\beta_j)_{t^2}, \quad \alpha_j, \beta_j \in \mathbb{N}, \quad (1)$$

but restricting the calculations to the special class of braided racks. With this approach, the Nichols algebras $\mathfrak{B}(Q_{3,1}, E_3)^{(2)}$ and $\mathfrak{B}(Q_{4,1}, \chi_4)$ were found to be finite-dimensional. Though the calculations are intricate already in the case of braided racks, the classification of finite-dimensional Nichols algebras along the approach proposed by Graña, Heckenberger and Vendramin seems to be feasible; if it is possible to show that the Hilbert series always factorizes in a way similar to Equation 1. This paper wants to contribute to the last question.

Shortly after publication of [15], Heckenberger brought to our attention that the dimensions of the Nichols algebras identified so far are always divisible by the order of the inner group of their underlying racks,

a fact which has been proven for Nichols algebras over Coxeter groups by Milinski and Schneider in 1999 ([17]). For their proof, they constructed a family of isomorphisms between the homogeneous components, based on interchanging the coefficients of a decomposition of each vector. This decomposition is possible due to the Nichols-Zoeller-Theorem ([20], see also Theorem 7.2.9 in [7]), which essentially states that a finite-dimensional Hopf algebra H is a free left B -module for all Hopf subalgebras B of H .

Similar freeness theorems are abundant in Hopf theory and can be found e.g. in [5] (Theorem 1 and Lemma 3.2), [22] (Theorem 6.1), and [21]. We will use a version by Graña:

Theorem 1 (Graña) **1**
(Cf. [9], Theorem 3.8.1) Let V be a finite-dimensional Yetter-Drinfeld module over a group G and assume $V = V' \oplus U$ with

- $V' \subseteq V$ a G' -stable $\mathbb{K}G$ -subcomodule, where G' is the smallest subgroup of G with $\delta(V') \subseteq \mathbb{K}G' \otimes V'$, and
- $U \subseteq V$ a $\mathbb{K}G$ -subcomodule and $\mathbb{K}G'$ -submodule.

Let $\{e_i : i = 1, \dots, d\}$ be a basis of V' and ∂_i the corresponding braided (skew) derivatives on the Nichols algebra $\mathfrak{B}(V)$ over V . Then

$$\mathfrak{B}(V) \cong \left(\bigcap_{i=1}^d \ker \partial_i \right) \otimes \mathfrak{B}(V')$$

as right $\mathfrak{B}(V')$ -modules and left $(\bigcap_{i=1}^d \ker \partial_i)$ -modules. In particular, $\dim \mathfrak{B}(V')$ divides $\dim \mathfrak{B}(V)$.

Applying this together with specialized maps similar to those used by Milinski and Schneider, our main results are as follows:

Theorem 2 **2**
Let \mathfrak{B} be a finite-dimensional Nichols algebra over the indecomposable quandle X with a 2-cocycle χ . Assume that the degree of X divides the order of the diagonal elements of χ . Then each $\text{Inn } X$ -homogeneous component of \mathfrak{B} has the same dimension (this is wrong for $\deg X \nmid \text{ord } \chi$ in general) and thus $\# \text{Inn } X$ divides $\dim \mathfrak{B}$.

Moreover, let X' be a non-empty proper sub-rack of X and \mathfrak{B}' the Nichols sub-algebra generated by X' . Assume that $X \setminus X'$ still generates $\text{Inn } X$. Then $\# \text{Inn } X \cdot \dim \mathfrak{B}'$ divides $\dim \mathfrak{B}$.

If we drop the assumption that the degree of X divides the order of the diagonal elements of χ , we can still prove the following:

Theorem 3 **3**
Let \mathfrak{B} be a finite-dimensional Nichols algebra over an indecomposable rack X and a 2-cocycle with diagonal elements of order m . Let X' be a non-empty proper subrack of X and \mathfrak{B}' its corresponding Nichols sub-algebra of \mathfrak{B} . Then the Hilbert series $\mathcal{H}_{\mathfrak{B}}(t)$ is divisible by $(m)_t \cdot \mathcal{H}_{\mathfrak{B}'}(t)$.

Our paper is organized as follows. In Section 2, we define the core notions to understand the main results (racks, quandles, Nichols algebras, (opposite) braided derivations, Hilbert series) as well as a list of known Nichols algebras of non-abelian group type with absolutely irreducible Yetter-Drinfeld modules and the corresponding quandles in Subsection

2.3. In Section 3, we introduce maps, so-called shifts, which are direct generalisations of the maps defined by Milinski and Schneider in [17]. An even more generalized form is currently considered by Angiono, Vay, and Vendramin. In Subsection 3.1, we apply them to show the first half of Theorem 2, i.e. $\# \text{Inn } X \mid \dim \mathfrak{B}$ for a slightly larger class of Nichols algebras than those examined in [17]. In Section 4, we define two improved variations of shifts and apply them to Graña's Freeness Theorem to show the second half of Theorem 2 for the same class of Nichols algebras as before. We will then use our methods to analyze arbitrary Nichols algebras over non-trivial, indecomposable racks and proof Theorem 3. Finally, we will show in Subsection 5.2, that the direct approach to proof $\# \text{Inn } X \mid \dim \mathfrak{B}$ along the lines of [17] and Subsection 3.1 is not feasible in the general case $\deg X \nmid \text{ord } \chi$.

2 Preliminaries

2.1 Notations

Denote $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$. With C_k we denote the cyclic group of order k , and with $[j]_k$ the equivalence class of j in C_k , for $k \in \mathbb{N}$ and $j \in \mathbb{Z}$. We use \mathfrak{S}_n to refer to the symmetric group on n symbols. Furthermore, we use the notation " $Q_{x,y}$ " to refer to the y -th indecomposable quandle of size x in Graña's and Vendramin's list of small indecomposable quandles ([24], implemented in Rig, see [12]). \mathbb{K} shall always be a field of arbitrary characteristic, if not said otherwise.

2.2 Nichols Algebras from Racks

For a detailed account on racks in the context of Nichols Algebras, see [1].

Definition 4 4

A rack X is a set with a binary operation \triangleright , which fulfills:

- *Left-self-distributivity:* For all $t_1, t_2, t_3 \in X$ holds $t_1 \triangleright (t_2 \triangleright t_3) = (t_1 \triangleright t_2) \triangleright (t_1 \triangleright t_3)$.
- *The operations $g_t : X \rightarrow X, s \mapsto t \triangleright s$ are bijections.*

An idempotent rack is called quandle.

Due to left-self-distributivity, each g_t as defined above is an automorphism of (X, \triangleright) . The permutation subgroup generated by the g_t is called the inner group $\text{Inn } X$ of X . It is a quotient of the enveloping (or structure) group

$$\text{Env } X := \langle t \in X \mid s \cdot t = (s \triangleright t) \cdot s \ \forall s, t \in X \rangle_{\text{group}}.$$

A rack X is called indecomposable, if $\text{Inn } X$ acts transitively on X . If the map $g : X \rightarrow \text{Inn } X, t \mapsto g_t$ is injective, X is called faithful.

If X is a faithful quandle, then X is realized as a conjugation-closed generating subset of a group. On the other hand, each such subset is a faithful quandle.

Throughout this article, let X denote a finite indecomposable faithful quandle and \mathbb{K} our ground-field.

Example 5 5

(Cf. [1], subsection 1.3) Let A be an abelian group and $\alpha : A \rightarrow A$ some automorphism of A . Then $\triangleright : A \times A \rightarrow A$, $(t, s) \mapsto \alpha(t - s) + s$ defines a quandle structure on A . Quandles of this kind are called affine quandles. Many of the quandles used to construct Nichols algebras are affine (see the tables in subsection 2.3), though not all. Assume A is an affine quandle with two commuting elements $t, s \in A$ (i.e. $t \triangleright s = s$ and vice versa), then inserting into the definition gives $t = s$. This excludes many quandles, e.g. the quandles given by transpositions in the symmetric group S_n for $n \geq 4$. The smallest non-affine quandles without commuting elements are $Q_{15,5}$ and $Q_{15,6}$ ([12]). On the other hand, there are many non-faithful affine quandles.

Definition 6 6

(Cf. [17]; for terms of Hopf algebra theory, see e.g. [7]) Let V_0 be a finite-dimensional vector space and set $V = \mathbb{K}X \otimes V_0$. Denote $V_t := \{t\} \otimes V_0$. Assume the inner group $\text{Inn } X$ acts on V with $g_t(V_s) \subseteq V_{t \triangleright s}$ for all $t, s \in X$. Then V is an $\text{Inn } X$ -Yetter-Drinfeld module, and a braiding

$$c(v \otimes w) = g_t(w) \otimes v$$

on the tensor product $V \otimes V$ is induced, where $v \in V_t$ and $w \in V$ are arbitrary. This in turn induces a co-algebra structure on the tensor algebra (TV, μ) , which is uniquely determined by the two requirements

1. $\Delta \mu = (\mu \otimes \mu)(\text{id}_V \otimes c \otimes \text{id}_V)(\Delta \otimes \Delta)$
2. and each $v \in V \subseteq TV$ is primitive.

The co-unit is given by $\epsilon(1) := 1$ and $\epsilon(v) := 0$ for all $v \in V \subseteq TV$. Furthermore, an antipode $S : TV \rightarrow TV$ can be defined which endows TV with the structure of an \mathbb{N}_0 -graded Hopf algebra (actually F_X -graded, where F_X is the free group over the set X).

There is a unique maximal homogeneous ideal and coideal I of TV such that

$$\Delta(I) \subseteq I \otimes TV + TV \otimes I$$

and such that all homogeneous elements of I are of degree ≥ 2 . The quotient $\mathfrak{B} := TV/I$ is called the Nichols algebra of V . It is an $\text{Env } X$ -graded braided Hopf algebra, whose primitive elements are exactly those of degree 1 and generate \mathfrak{B} .

The Nichols algebra \mathfrak{B} can be completely described in terms of the rack X and a 2-cocycle $\chi : X \times X \rightarrow \text{End}(V_0)$ satisfying

$$\chi(t, s \triangleright r) \chi(s, r) = \chi(t \triangleright s, t \triangleright r) \chi(t, r),$$

which induces the action of $\text{Inn } X$ on V ([1], [3]).

If V is finite-dimensional and absolutely irreducible as a Yetter-Drinfeld module, the Lemma of Schur shows that $\chi(t, t)$ actually is a scalar multiple q_t of the identity for each $t \in X$ ([11]; Theorem 2.7 in [10]). If X is indecomposable, the transitive action of $\text{Inn } X$ ensures that q_t does not depend on t ; we drop the index in this case. However, Graña showed in [10], Lemma 3.1, that the cases $\dim V_0 \geq 2$ impose severe restrictions on q and χ if \mathfrak{B} is to be finite-dimensional. Therefore, for the most part of this paper, we will restrict to the case $\dim V_0 = 1$, without losing too many cases.

Definition 7 7

Let X be an indecomposable rack. Then the order of g_t does not depend on $t \in X$ and we define the degree $\deg X$ to be this number.

The nilpotency order $\text{nord } v$ of an element $v \in \mathfrak{B}$ is the minimal $m \in \mathbb{N}$ with $v^m = 0$.

The order $\text{ord } \chi$ of a 2-cocycle χ is the minimal $m \in \mathbb{N} \cup \{\infty\}$ with $\chi(t, t)^m = 1$ for all $t \in X$. (If \mathfrak{B} is finite dimensional, one easily shows that $q_t := \chi(t, t)$ has to be a root of unity, such that $\text{ord } \chi$ is finite.)

For the rest of this paper, set $n := \deg X$ and $m := \text{ord } \chi$.

Denote with $\{e_t\}_{t \in X}$ the standard base in $\mathbb{K}X$. If V_0 is one-dimensional, we use it as standard base for V as well; else, we denote $e_t \otimes v$ with $e_t v$ (see [3]). If X is indecomposable, $\text{nord}(e_t v)$ does not depend on $t \in X$ nor $v \in V_0$. Using Equation 9 in Proposition 11 one easily calculates $\text{nord}(e_t v) = m$ in this case for $v \in V_0 \setminus \{0\}$. There is however no obvious relation between m and n , as can be seen in the examples of subsection 2.3.

Definition 8 8

Given a G -graded vector space U , denote with $U(g)$ the g -homogeneous subspace of U . We say the grading is balanced, if $\dim U(g)$ is finite and constant for all $g \in G$. A vector space U is G -balanced, if U is G -graded and the grading is balanced.

Let U be a G -graded vector space and H a quotient of G . Then U is H -graded as well. Any Nichols algebra \mathfrak{B} is $\text{Env } X$ -graded. As there are canonical surjective homomorphisms $\text{Env } X \rightarrow \mathbb{Z}$ and $\text{Env } X \rightarrow \text{Inn } X$, we will use the notation $\mathfrak{B}(x)$ for any $x \in \text{Env } X$, \mathbb{Z} or $\text{Inn } X$ without further notice, as the latter two gradings are induced by the first one. If the G -grading is balanced, then the induced H -grading is balanced as well.

Definition 9 9

If U is a \mathbb{Z} -graded vector space and each $U(g)$ is finite-dimensional, define the (formal) Hilbert series \mathcal{H}_U by

$$\mathcal{H}_U(t) := \sum_{j \in \mathbb{Z}} \dim U(j) t^j.$$

For any $k \in \mathbb{N} \cup \{\infty\}$ define $(k)_t$ to be the series $\sum_{j=1}^{k-1} t^j$.

Finite dimensional Nichols algebras of abelian group type (i.e. over trivial quandles) have been completely classified by I. Heckenberger in [13] and [14]. The classification of finite dimensional Nichols algebras of non-abelian group type, particularly over indecomposable quandles, is advanced by several strategies. A very interesting ansatz is to identify the set of appropriate racks, so by identifying racks of type D. This led to the exclusion of conjugacy class racks of whole classes of groups, notably the alternating groups A_m for $m \geq 6$ ([2]) and many sporadic groups ([4]). An alternative is to derive inequalities on the maximal dimensions of the lower homogeneous degrees, as has been done in [11] and [15], and connect these to a certain factorization of $\mathcal{H}_{\mathfrak{B}}$ in terms of $(k)_t$ and $(k)_{t^2}$.

2.3 Examples for Nichols Algebras

There are nine indecomposable and faithful quandles known to provide examples of finite-dimensional Nichols algebras, with the following properties.

X	$\deg X$	$\text{Inn } X$	(Size)	$g_t \in \text{Inn } X$ for generating $t \in X$ (as perm. of X in cycle notation)
$Q_{3,1}$	2	S_3	(6)	$g_1 = (2, 3), g_2 = (1, 3)$
$Q_{4,1}$	3	A_4	(12)	$g_1 = (2, 3, 4), g_2 = (1, 4, 3)$
$Q_{5,2}$	4	$C_5 \rtimes C_4$	(20)	$g_1 = (2, 4, 5, 3), g_2 = (1, 4, 3, 5)$
$Q_{5,3}$	4	$C_5 \rtimes C_4$	(20)	$g_1 = (2, 3, 5, 4), g_2 = (1, 5, 3, 4)$
$Q_{6,1}$	2	S_4	(24)	$g_1 = (3, 5)(4, 6), g_2 = (3, 6)(4, 5),$ $g_3 = (1, 5)(2, 6)$
$Q_{6,2}$	4	S_4	(24)	$g_1 = (3, 5, 4, 6), g_3 = (1, 6, 2, 5)$
$Q_{7,4}$	6	$(C_7 \rtimes C_3) \rtimes C_2$	(42)	$g_1 = (2, 6, 5, 7, 3, 4),$ $g_2 = (1, 4, 5, 3, 7, 6)$
$Q_{7,5}$	6	$(C_7 \rtimes C_3) \rtimes C_2$	(42)	$g_1 = (2, 4, 3, 7, 5, 6),$ $g_2 = (1, 6, 7, 3, 5, 4)$
$Q_{10,1}$	2	S_5	(120)	$g_1 = (2, 7)(3, 5)(4, 6),$ $g_2 = (1, 7)(3, 8)(4, 10),$ $g_3 = (1, 5)(2, 8)(4, 9),$ $g_4 = (1, 6)(2, 10)(3, 9)$

The quandles $Q_{3,1}$, $Q_{5,2}$, $Q_{5,3}$, $Q_{7,4}$, and $Q_{7,5}$ are affine quandles over the cyclic abelian groups of order $\#X$, with α the multiplication with 2, 3, 2, 5, and 3, respectively. $Q_{4,1}$ also is an affine quandle over $C_2 \times C_2$ with $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The quandles $Q_{3,1}$, $Q_{6,1}$ and $Q_{10,1}$ can also be defined as the conjugacy classes of transpositions in the symmetric groups \mathfrak{S}_n for $n = 3, 4, 5$, respectively.

The following fourteen finite-dimensional Nichols algebras over \mathbb{K} of non-abelian group type are our basic examples, sorted by quandle and dimension.

$\mathfrak{B}(X, c)$	$\text{char}(\mathbb{K})$	n	m	dimension	$\mathcal{H}_{\mathfrak{B}}(t)$
$\mathfrak{B}(Q_{3,1}, -1)$	*	2	2	12	$(2)_t^2 (3)_t$
$\mathfrak{B}(Q_{3,1}, E_3)^{(2)}$	2	2	3	432	$(3)_t (4)_t (6)_t (6)_{t^2}$
$\mathfrak{B}(Q_{4,1}, -1)^{(2)}$	2	3	2	36	$(2)_t^2 (3)_t^2$
$\mathfrak{B}(Q_{4,1}, -1)^{(\neq 2)}$	$\neq 2$	3	2	72	$(2)_t^2 (3)_t (6)_t$
$\mathfrak{B}(Q_{4,1}, \chi_4)$	*	3	3	5,184	$(6)_t^4 (2)_{t^2}^2$
$\mathfrak{B}(Q_{5,2}, -1)$	*	4	2	1,280	$(4)_t^4 (5)_t$
$\mathfrak{B}(Q_{5,3}, -1)$	*	4	2	1,280	$(4)_t^4 (5)_t$
$\mathfrak{B}(Q_{6,1}, -1)$	*	2	2	576	$(2)_t^2 (3)_t^2 (4)_t^2$
$\mathfrak{B}(Q_{6,1}, \chi_6)$	*	2	2	576	$(2)_t^2 (3)_t^2 (4)_t^2$
$\mathfrak{B}(Q_{6,2}, -1)$	*	4	2	576	$(2)_t^2 (3)_t^2 (4)_t^2$
$\mathfrak{B}(Q_{7,4}, -1)$	*	6	2	326,592	$(6)_t^6 (7)_t$
$\mathfrak{B}(Q_{7,5}, -1)$	*	6	2	326,592	$(6)_t^6 (7)_t$
$\mathfrak{B}(Q_{10,1}, -1)$	*	2	2	8,294,400	$(4)_t^4 (5)_t^2 (6)_t^4$
$\mathfrak{B}(Q_{10,1}, \chi_{10})$	*	2	2	8,294,400	$(4)_t^4 (5)_t^2 (6)_t^4$

In all cases we have $\dim V_0 = 1$. E_N denotes an N -th root of unity and the superscripts $^{(2)}$ and $^{(\neq 2)}$ refer to the field's characteristic. The

non-constant cocycles χ_4 , χ_6 and χ_{10} are defined as follows:

$$\chi_4 := \begin{pmatrix} E_3 & -E_3 & -E_3 & E_3 \\ -E_3 & E_3 & -E_3 & E_3 \\ -E_3 & -E_3 & E_3 & E_3 \\ E_3 & E_3 & E_3 & E_3 \end{pmatrix}$$

$$\chi_6 := \begin{pmatrix} -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 1 & -1 \end{pmatrix}$$

$$\chi_{10} := \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \end{pmatrix}$$

A more concise description of χ_6 and χ_{10} in terms of transpositions in \mathfrak{S}_4 and \mathfrak{S}_5 is given e.g. in [17] (Example 5.3) and in [23].

Note that $\mathfrak{B}(Q_{5,2}, -1)$ and $\mathfrak{B}(Q_{5,3}, -1)$ as well as $\mathfrak{B}(Q_{7,4}, -1)$ and $\mathfrak{B}(Q_{7,5}, -1)$ are dual algebras (see Example 2.1 in [3]), $\mathfrak{B}(Q_{6,1}, -1)$ and $\mathfrak{B}(Q_{6,1}, \chi_6)$ as well as $\mathfrak{B}(Q_{10,1}, -1)$ and $\mathfrak{B}(Q_{10,1}, \chi_{10})$ are twist-equivalent to each other ([23]).

Also note that the factorization of $\mathcal{H}_{\mathfrak{B}}(t)$ in terms of $(k)_t$ and $(k)_{t^2}$ is not unique.

2.4 Braided Derivations and Braided Commutator

For the rest of the paper, we assume $\dim V_0 = 1$ for simplicity.

Definition 10

Given the comultiplication $\Delta : \mathfrak{B} \rightarrow \mathfrak{B}$, we can uniquely define linear maps ∂_t and $\partial_t^{\text{op}} : \mathfrak{B} \rightarrow \mathfrak{B}$ for arbitrary $t \in X$ via

$$\begin{aligned} \Delta(v) &= v \otimes 1 + \sum_{t \in X} \partial_t(v) \otimes e_t + \text{some element of } \mathfrak{B} \otimes \bigoplus_{j=2}^{\infty} \mathfrak{B}(j) \\ &= 1 \otimes v + \sum_{t \in X} e_t \otimes \partial_t^{\text{op}}(v) + \text{some element of } \bigoplus_{j=2}^{\infty} \mathfrak{B}(j) \otimes \mathfrak{B}. \end{aligned}$$

We call these maps braided derivations and opposite braided derivations, respectively.

The braided derivations ∂ and ∂^{op} , have been introduced by Nichols in subsection 3.3 in [18] under the name “quantum differential operators”; for an account on them, we refer to [5].

Proposition 11 **11**

The maps ∂_t and ∂_t^{op} satisfy the following properties for all $t, s \in X$ and $v, w \in \mathfrak{B}$:

$$\partial_t(1) = 0 \quad (2)$$

$$\partial_t(e_s) = \delta_{t,s} \quad (3)$$

$$\partial_t(vw) = v\partial_t(w) + \partial_t(v)g_t(w) \quad (4)$$

$$\partial_t^{\text{op}}(1) = 0 \quad (5)$$

$$\partial_t^{\text{op}}(e_s) = \delta_{t,s} \quad (6)$$

$$\partial_t^{\text{op}}(vw) = \partial_t^{\text{op}}(v)w + v\partial_{g_v^{-1}(t)}^{\text{op}}(w)/\chi(v,t) \quad (\text{if } v \text{ is homog.}) \quad (7)$$

$$\partial_s^{\text{op}}\partial_t = \partial_t\partial_s^{\text{op}} \quad (8)$$

$$\bigcap_{t \in X} \ker \partial_t = \bigcap_{s \in X} \ker \partial_s^{\text{op}} = \mathfrak{B}(0) \quad (9)$$

(where $\delta_{t,s}$ is the Kronecker symbol.)

Proof (2), (5): Obvious.

(3), (6): Follows from primitivity of $\mathfrak{B}(1)$.

(8): Follows from co-associativity.

(9): Follows from [18] and [9].

(4): Let $v, w \in \mathfrak{B}$ be arbitrary. Then holds:

$$\begin{aligned} \Delta(vw) &= \Delta(v) \cdot \Delta(w) \\ &= \left(v \otimes 1 + \sum_{t \in X} \partial_t(v) \otimes e_t + \text{higher terms} \right) \\ &\quad \cdot \left(w \otimes 1 + \sum_{t \in X} \partial_t(w) \otimes e_t + \text{higher terms} \right) \\ &= (vw) \otimes 1 + \sum_{t \in X} (\partial_t(v) \otimes e_t) \cdot (w \otimes 1) \\ &\quad + \sum_{t \in X} (v \otimes 1) \cdot (\partial_t(w) \otimes e_t) + \text{higher terms} \\ &= (vw) \otimes 1 + \sum_{t \in X} (\partial_t(v)g_t(w) + v\partial_t(w)) \otimes e_t + \text{h.t.} \end{aligned}$$

(7): Similar to (4) we have:

$$\begin{aligned} \Delta(vw) &= \Delta(v) \cdot \Delta(w) \\ &= \left(1 \otimes v + \sum_{t \in X} e_t \otimes \partial_t^{\text{op}}(v) + \text{higher terms} \right) \\ &\quad \cdot \left(1 \otimes w + \sum_{s \in X} e_s \otimes \partial_s^{\text{op}}(w) + \text{higher terms} \right) \\ &= 1 \otimes (vw) + \sum_{t \in X} (e_t \otimes \partial_t^{\text{op}}(v)) \cdot (1 \otimes w) \\ &\quad + \sum_{s \in X} (1 \otimes v) \cdot (e_s \otimes \partial_s^{\text{op}}(w)) + \text{higher terms} \\ &= 1 \otimes (vw) + \sum_{t \in X} (e_t \otimes \partial_t^{\text{op}}(v)w) + \sum_{s \in X} (g_v(e_s) \otimes v\partial_s^{\text{op}}(w)) + \text{h.t.} \end{aligned}$$

where g_v is defined such that $v \in \mathfrak{B}(g_v)$. By definition we have $g_v(e_s) = \chi(v, s) e_{v \triangleright s}$ (where $v \triangleright s$ is short-hand for $g_v(s)$ and the 2-cocycle χ is extended in the obvious way). Choosing t such that $e_{v \triangleright s} = e_t$, we conclude

$$\Delta(vw) = 1 \otimes (vw) + \sum_{t \in X} e_t \otimes (\partial_t^{\text{op}}(v)w + v \partial_{v \triangleright^{-1}t}^{\text{op}}(w)/\chi(v, t)) + \text{h.t.}$$

where $v \triangleright^{-1}t := g_v^{-1}(t)$. \square

∂_t is a right σ -skew-derivation, as one sees from Equation 4, with the endomorphism $\sigma = g_t$. ∂_t^{op} is not a right skew-derivation; so we chose the word “opposite braided derivation”, to emphasize its kinship with ∂_t . Also note that $\ker \partial_t^{\text{op}} \neq \ker \partial_t$ in general.

Definition 12 ([6]) **12**

Let \mathfrak{B} be a Nichols algebra with braiding c . Define the braided commutator

$$[x, y]_c := \mu \circ (\text{id} - c)(x \otimes y)$$

for all $x, y \in \mathfrak{B}$.

Proposition 13 **13**

For all $t \in X$ holds: $\partial_t g_t = g_t \cdot g_t \partial_t$.

Proof By induction over the \mathbb{N}_0 -degree d of $v \in \mathfrak{B}$. For $d \in \{0, 1\}$, this is clear. For each $v, w \in \mathfrak{B}$ we have

$$\begin{aligned} \partial_t g_t(vw) &= (g_t v) \cdot (\partial_t g_t w) + (\partial_t g_t v) \cdot (g_t^2 w) \\ &\stackrel{\text{ind.}}{=} g_t \cdot ((g_t v) \cdot (g_t \partial_t w) + (g_t \partial_t v) \cdot (g_t^2 w)) \\ \text{and } g_t \partial_t(vw) &= (g_t v) \cdot (g_t \partial_t w) + (g_t \partial_t v) \cdot (g_t^2 w). \end{aligned}$$

\square

Proposition 14 **14**

Let $t \in X$ and $v \in \ker \partial_t$ be arbitrary. Then $[e_t, v]_c \in \ker \partial_t$.

Proof With Proposition 13 one finds

$$\partial_t([e_t, v]_c) = \partial_t(e_t v) - \partial_t((g_t v)e_t) = g_t v - g_t v = 0.$$

\square

3 The Shift Group of a Nichols Algebra

Milinski and Schneider showed in [17], Theorem 5.8, that the grading of a Nichols algebra is balanced, if $\text{Inn } X$ is a Coxeter group and a certain type of cocycle is given. In general, the grading of a Nichols algebra need not be balanced, as we will see in the case of the 72-dimensional Nichols algebra in subsection 5.2.

Let \mathfrak{B} be a finite-dimensional Nichols-Algebra over the rack X and cocycle χ with $\dim V_0 = 1$ (see Definition 6). Recall that \mathfrak{B} is generated as an algebra by the elements $e_t \in \mathfrak{B}(1)$, $t \in X$.

It is a well-known fact that for each finite-dimensional Nichols algebra \mathfrak{B} over the quandle X and the field \mathbb{K} and for each $t \in X$ holds:

Lemma 15 15

Let $t \in X$ be arbitrary and $m = \text{nord } e_t$. Then holds

$$\mathfrak{B} \cong (\ker \partial_t) \otimes (\mathbb{K}[e_t]/e_t^m) \quad \text{and} \quad \mathfrak{B} \cong (\mathbb{K}[e_t]/e_t^m) \otimes (\ker \partial_t^{\text{op}}).$$

Proof 1) This directly follows from Graña's Freeness Theorem. We reproduce a short proof to compare it to part (2).

Let $s \in X \setminus \{t\}$ be arbitrary. Then by definition of the braided commutator holds:

$$e_t e_s = [e_t, e_s]_c + (g_t e_s) e_t.$$

For $[e_t, e_s]_c$ we use Proposition 14 to see that $[e_t, e_s]_c \in \ker \partial_t$. On the other hand, we have $\partial_t(g_t e_s) = 0$, because $t \triangleright s \neq t$ for $s \neq t$ and any quandle X . By induction we find that for each $v \in \mathfrak{B}$ there are $v_j \in \ker \partial_t$ with

$$v = \sum_{j=0}^{m-1} v_j e_t^j.$$

We now show that these v_j are uniquely determined (this is analog to Lemma 2.5 in [17]): Assume $\sum_{j=0}^{m-1} v_j e_t^j = 0$. Apply ∂_t $(m-1)$ -times to find $v_{m-1} = 0$. Then apply ∂_t $(m-2)$ -times to see $v_{m-2} = 0$, induction.

2) Let $v \in \mathfrak{B}$ be homogeneous with $\partial_t^{\text{op}}(v) \neq 0$. By induction over the length, we can restrict to $v = u e_s$ with $\partial_t^{\text{op}}(u) = 0$ but $\partial_t^{\text{op}}(u e_s) \neq 0$. Set $w := \chi(u, t)^{-1} u$. Then $w \in \ker \partial_t^{\text{op}}$ and

$$\underbrace{\partial_t^{\text{op}}(u e_s)}_{=v} - e_t w = \chi(u, t)^{-1} u \underbrace{\partial_{u \triangleright^{-1} t}^{\text{op}} e_t}_{=1} - w - q_t^{-1} e_t \partial_t^{\text{op}} w = 0,$$

hence $v = e_t w - (v - e_t w) \in (\mathbb{K}[e_t]/e_t^m) \otimes (\ker \partial_t^{\text{op}})$. Like in (1), linear independence is shown by applying ∂_t^{op} . \square

One would expect that for each element $v \in \mathfrak{B}(1)$, there is a decomposition $\mathfrak{B} = U \otimes \mathbb{K}[v]/v^{\text{nord } v}$ similar to the one of Lemma 15. This, however, is wrong: Take $v = e_1 + e_2 \in \mathfrak{B}(Q_{3,1}, -1)$. If K is of characteristic $\neq 2$, v has nilpotency order 4. If $\mathfrak{B}(Q_{3,1}, -1)$ would decompose into a tensor product with factor $\mathbb{K}[v]/v^4$, its Hilbert series would be divisible by $(4)_t$, which is not the case.

Proposition 16 16

Let $v \in \mathfrak{B}(g)$ for some $g \in \text{Env } X$ and $t \in X$ be arbitrary. If we decompose v into the sum $\sum_{j=0}^{m-1} v_j e_t^j$ with Lemma 15, we have $v_j \in \mathfrak{B}(g t^{-j})$ for each j . If decomposed into $v = \sum_{j=0}^{m-1} e_t^j v_j$, each $v_j \in \mathfrak{B}(t^{-j} g)$.

Proof Each summand $v_j e_t^j$ is itself $\text{Env } X$ -homogeneous (otherwise there would be a non-trivial linear dependency). Due to uniqueness, all $v_j e_t^j$ are linearly independent, hence each $v_j e_t^j$ must be element of $\mathfrak{B}(g) \ni v$. Then $v_j \in \mathfrak{B}(h)$ with $h = g t^{-j}$. The second statement follows the same way. \square

Proposition 17 17

Let $t, s \in X$ be arbitrary, $t \neq s$. The shift

$$\begin{aligned} \phi_t : \mathfrak{B} &\rightarrow \mathfrak{B}, & v &= \sum_{j=0}^{m-1} v_j e_t^j \mapsto v_{m-1} + \sum_{j=1}^{m-1} v_{j-1} e_t^j \\ && \forall j : v_j &\in \ker \partial_t \end{aligned}$$

is a well-defined linear isomorphism. We call the group generated by the shifts ϕ_t the shift group $\Phi(\mathfrak{B})$ (or just Φ). By definition, its operation on \mathfrak{B} is free.

Proof ϕ_t is bijective because $\phi_t^m = \text{id}_{\mathfrak{B}}$ and ϕ_t obviously is linear. \square

Lemma 18 18
The Φ -orbit of 1 linearly spans \mathfrak{B} .

Proof By induction over the \mathbb{N}_0 -degree d of $v \in \mathfrak{B}$. For $d \in \{0, 1\}$, this is clear. Assume $\Phi(1)$ spans the whole of $\mathfrak{B}(d)$. Let $w \in \mathfrak{B}(d)$ and $t \in X$ be arbitrary. By Lemma 15, w decomposes into

$$w = \sum_{j=0}^{m-1} w_j e_t^j$$

with $w_j \in \ker \partial_t$, $j = 0 \dots (m-1)$. Due to the grading, each w_j can be chosen to be of length $d-j$. Now we see

$$w e_t = \sum_{j=0}^{m-2} w_j e_t^{j+1} = \phi_t \left(\sum_{j=0}^{m-2} w_j e_t^j \right).$$

$\sum_{j=0}^{m-2} w_j e_t^j$ is of length d and can be spanned by $\Phi(1)$. Hence, $w e_t$ can be spanned by $\Phi(1)$ as well. \square

As an (unused) corollary, we see that if Φ is finite, \mathfrak{B} must be finite-dimensional. However, the converse is not true: $\Phi(\mathfrak{B}(Q_{4,1}, -1)^{(\neq 2)})$ contains the infinite group $C_2 * C_2$ as a subgroup (see subsection 5.2).

The dimension of the matrix algebra $\text{Alg } \Phi$ spanned by the maps ϕ_t is bounded from above by $(\dim \mathfrak{B})^2$; and if \mathfrak{B} is infinite, it must be infinite-dimensional as well, due to Lemma 18. In the case of the Nichols algebra $\mathfrak{B}(Q_{3,1}, -1)$, we find that the dimension of this shift algebra is 12, which equals the dimension of \mathfrak{B} . But we have $\phi_t^2 = \text{id}$ in this algebra, so it cannot be \mathbb{N}_0 -graded. Hence, the shift algebra and \mathfrak{B} are not isomorphic.

In the case of $\mathfrak{B}(Q_{4,1}, -1)^{(2)}$, even the dimensions differ: \mathfrak{B} has dimension 36, but the shift algebra of \mathfrak{B} has dimension $648 = 18 \cdot 36$. For the 72-dimensional algebra $\mathfrak{B}(Q_{4,1}, -1)^{(\neq 2)}$ one finds that the shift algebra has dimension $2592 = 36 \cdot 72$.

3.1 Case n divides m

In the following, let \mathfrak{B} be a Nichols algebra with $n \mid m$, Φ its shift group and $\mathbb{K}\Phi$ the group algebra of Φ . (Instead of $\mathbb{K}\Phi$ we might just as well take the shift algebra of \mathfrak{B} .)

The evaluation at 1 $\in \mathfrak{B}$ yields a linear map $\text{ev}_1 : \mathbb{K}\Phi \rightarrow \mathfrak{B}$, which by Lemma 18 is surjective. In particular, we may define subspaces $\mathbb{K}\Phi_g := \text{ev}_1^{-1}(\mathfrak{B}(g))$.

Proposition 19 19

If ϕ_t is restricted to $\mathfrak{B}(g)$ for any $g \in \text{Inn } X$, its image restricts to $\mathfrak{B}(gt)$ and hence yields a linear isomorphism $\mathfrak{B}(g) \cong \mathfrak{B}(gt)$.

Proof We use the notations from Proposition 17. Let $v \in \mathfrak{B}(g)$. Then $v_j \in \mathfrak{B}(gt^{-j})$ and hence $v_{j-1} e_t^j \in \mathfrak{B}(gt)$ for each $j = 1 \dots (m-1)$. Because of $n \mid m$ we have $t^m = 1$ in $\text{Inn } X$ and $v_{m-1} \in \mathfrak{B}(h)$ with $h = gt^{-(m-1)} = gt$, so the sum $\phi_t(v)$ is in $\mathfrak{B}(gt)$. Apply ϕ_t m -times to see that $\phi_t|_{\mathfrak{B}(g)} : \mathfrak{B}(g) \rightarrow \mathfrak{B}(gt)$ is surjective and hence an isomorphism. \square

Corollary 20 20
The grading of \mathfrak{B} is balanced. The order of $\text{Inn } X$ divides $\dim \mathfrak{B}$.

Proof X generates $\text{Inn } X$. \square

In particular, we see that $\mathbb{K}\Phi_g \cdot \mathbb{K}\Phi_h \subseteq \mathbb{K}\Phi_{gh}$, so $\mathbb{K}\Phi / \ker \text{ev}_1$ has a G -grading—it actually is just the G -grading of \mathfrak{B} pushed to $\mathbb{K}\Phi / \ker \text{ev}_1 \cong \mathfrak{B}$.

Lemma 21 21
 $\text{Inn } X$ is a quotient of Φ .

Proof By Proposition 19, the map $\gamma := \deg \circ \text{ev}_1 : \Phi \rightarrow G$ is a surjective homomorphism. \square

Given a subset X' of X , let \mathfrak{B}' be the subalgebra-with-one of \mathfrak{B} generated by $\mathbb{K}X' \subseteq \mathfrak{B}(1)$. If X' is a subrack of X , then \mathfrak{B}' is its corresponding Nichols subalgebra.

Proposition 22 22
Let X' be a subrack of X and $v \in \mathfrak{B}'$.

- 1) *If $\partial_t v = 0$ for all $t \in X'$, then $v \in \mathfrak{B}(0)$.*
- 2) *If $\partial_t^{\text{op}} v = 0$ for all $t \in X'$, then $v \in \mathfrak{B}(0)$.*

Proof From $v \in \mathfrak{B}'$ we know $\partial_s v = 0 = \partial_s^{\text{op}} v$ for all $s \in X \setminus X'$, so v must be a multiple of 1. Note that (1) actually holds for arbitrary subsets X' of X when \mathfrak{B}' is defined as the sub-algebra generated by all e_t with $t \in X'$; this is not true for (2). \square

Lemma 23 23
Let X' be a non-empty proper subrack of X and G' the subgroup of $\text{Inn } X$ spanned by the operations $g_t : X \rightarrow X$ for all $t \in X'$. Let V' be the linear span of the elements e_t with $t \in X'$ and U the linear span of all e_s with $s \in X \setminus X'$. Then holds:

1. *G' is the smallest subgroup of $G = \text{Inn } X$ with $\delta(V') \subseteq \mathbb{K}G' \otimes V'$,*
2. *V' is G' -stable,*
3. *V' and U are $\mathbb{K}G$ -subcomodules, and*
4. *U is a $\mathbb{K}G'$ -submodule.*

In particular, Graña's Freeness Theorem (Theorem 1) applies and we find

$$\mathfrak{B} \cong \left(\bigcap_{t \in X'} \ker \partial_t \right) \otimes \mathfrak{B}',$$

where \mathfrak{B}' is the Nichols sub-algebra generated by X' .

Proof (1) holds by definition of G' . (2) holds because X' is a subrack. (3) is due to the diagonal comodule structure $\delta(e_t) = g_t \otimes e_t$ for all $t \in X$. To show (4), let $t \in X'$ and $s \in X \setminus X'$ be arbitrary. Then $g_t(e_s)$ is a multiple of $e_{t \triangleright s}$. The element $t \triangleright s$ cannot be in X' (otherwise s would be in X'), so $g_t(e_s) \in U$. \square

Freeness theorems as Grana's allow for recursion, such that \mathfrak{B} can be written as tensor product of terms of the form $\bigcap_{t \in X'} \ker \partial_t$ for ever decreasing subracks X' . Such a factorization induces a factorization of the Hilbert series as well. In the case of the related Fomin-Kirillov algebras, the analogous factorization has been conjectured in [16], Conjecture 8.6, and has been proven by Fomin and Procesi in [8]. Their factors are subalgebras generated by transpositions (i, n) for fixed n and $1 \leq i < n$. This corresponds to the intersection $\bigcap_{t \in X'} \ker \partial_t$ if X is the rack generated by the transpositions in \mathfrak{S}_n and X' the subrack generated by the transpositions of $\mathfrak{S}_{n-1} < \mathfrak{S}_n$. The subalgebra generated by $X \setminus X'$ is a subspace of $\bigcap_{t \in X'} \ker \partial_t$, but not necessarily all of it.

Proposition 24 **24**
Let X' be a non-empty subset of X . Let $G = \text{Env } X$ or any quotient of $\text{Env } X$. Then $\bigcap_{t \in X'} \ker \partial_t \subseteq \mathfrak{B}$ has a G -homogeneous basis.

Proof Each ∂_t is a G -graded map, hence $\ker \partial_t$ is a G -graded sub- \mathfrak{B} -module of \mathfrak{B} ; same for their intersection $\bigcap_{t \in X'} \ker \partial_t$. Each graded submodule has a homogeneous basis, see e.g. section 2.1 in [19]. \square

4 The Modified Shift Groups

Each shift ϕ_t can be written in the form

$$\phi_t(v) = \frac{1}{(1 + q_t)(1 + q_t + q_t^2) \cdots (1 + q_t + \cdots + q_t^{m-2})} \cdot \partial_t^{m-1} v + v e_t$$

by inserting the decomposition of v into the right-hand side. We modify this definition by removing the leading factor and subtracting the braided commutator for one variant; and by issuing the opposite braided derivative for another.

Definition 25 **25**
Let \mathfrak{B} be a Nichols algebra with rack X and $t \in X$ arbitrary. Define the modified shifts

$$\psi_t(v) := \partial_t^{m-1}(v) + e_t g_t^{-1}(v)$$

and

$$\xi_t(v) := (\partial_t^{\text{op}})^{m-1}(v) + e_t v$$

and the corresponding modified shift groups Ψ and Ξ , generated by all ψ_t (respectively ξ_t) with $t \in X$. If X' is a subset of X , define $\Psi|_{X'}$ and $\Xi|_{X'}$ to be the groups generated by all ψ_t (respectively ξ_t) with $t \in X'$.

Proposition 26 **26**
Let $s, t \in X$ and $g \in G$ be arbitrary, where G is any quotient of $\text{Env } X$. Then holds:

1. ψ_t is a linear isomorphism.

2. If $t^m = e$ in G , then ψ_t maps $\mathfrak{B}(g)$ to $\mathfrak{B}(gt)$.
3. If $t \neq s$, then $\psi_t(\ker \partial_s^{\text{op}}) = \ker \partial_s^{\text{op}}$.
4. The Ψ -orbit of 1 linearly spans \mathfrak{B} .

Proof 1. Linearity is obvious. Now assume $v \in \ker \psi_t \setminus \{0\}$. Then $\partial_t^{m-1}(v) = -e_t g_t^{-1}(v)$. Use Lemma 15 to decompose $v = \sum_{j=0}^{m-1} v_j e_t^j$ with $v_j \in \ker \partial_t$. Inserting this yields $v_{m-1} = \sum \lambda_j e_t g_t^{-1}(v_j) e_t^j$ for some $\lambda_j \in \mathbb{K} \setminus \{0\}$. Using the braided commutator, we find

$$e_t g_t^{-1}(v_j) e_t^j = v_j e_t^{j+1} + \underbrace{[e_t, g_t^{-1}(v_j)]_c}_{\in \ker \partial_t} e_t^j$$

by Propositions 13 and 14. We know $v_{m-1} \in \ker \partial_t$, and by comparing the coefficients of e_t^j , we find:

$$\begin{aligned} v_{m-1} &= \lambda_0 [e_t, g_t^{-1}(v_0)]_c \\ v_{j-1} &= -[e_t, g_t^{-1}(v_j)]_c \quad \text{if } 1 \leq j \leq m-1 \end{aligned}$$

In particular, the minimal length of the \mathbb{N}_0 -homogeneous components of v_{j-1} is at least one plus the minimal length of v_j , and the minimal length of v_{m-1} is at least one plus the minimal length of v_0 . This cannot be, hence all v_j are zero.

2. Let $v \in \mathfrak{B}(g) \setminus \{0\}$ be arbitrary. Then the degree of $\partial_t^{m-1}(v) = gt^{1-m} = gt$ and the degree of $e_t g_t^{-1}(v)$ also is $tt^{-1}gt = gt$. Thereby, $\psi_t(v)$ is homogeneous of degree gt .

3. Let $v \in \ker \partial_s^{\text{op}}$ be arbitrary. From Proposition 11 we conclude that $\partial_t^{m-1}(v) \in \ker \partial_s^{\text{op}}$. For the other summand, we find

$$\partial_s^{\text{op}}(e_t g_t^{-1}(v)) = q e_t \partial_{t \triangleright^{-1} s}^{\text{op}} g_t^{-1}(v).$$

v is in $\ker \partial_s^{\text{op}}$, so $g_t^{-1}(v)$ is in $\ker \partial_{t \triangleright^{-1} s}^{\text{op}}$.

4. We may use the same induction as in the proof of Lemma 18: We write

$$e_t v = \psi_t(g_t v) - \partial_t^{m-1}(g_t v)$$

and note that both summands on the right hand side are in the linear span of $\Phi(1)$. \square

Property (3) above is something not found in the shifts introduced in Proposition 17: In the Nichols algebra $\mathfrak{B}(Q_{3,1}, -1)$ choose $v = e_3$. Then $v \in \ker \partial_1^{\text{op}}$, but $\phi_2(v) = e_3 e_2 \notin \ker \partial_1^{\text{op}}$, while $\psi_2(v) = -e_2 e_1 \in \ker \partial_1^{\text{op}}$. This justifies the introduction of the modified shifts.

Proposition 27 **27**

Let $s, t \in X$ and $g \in G$ be arbitrary, where G is any quotient of $\text{Env } X$. Then holds:

1. ξ_s is a linear isomorphism.
2. If $s^m = e$ in G , then ξ_s maps $\mathfrak{B}(g)$ to $\mathfrak{B}(sg)$.
3. If $t \neq s$, then $\xi_s(\ker \partial_t) = \ker \partial_t$.
4. The Ξ -orbit of 1 linearly spans \mathfrak{B} .

Proof The proof is very similar to the proof of Proposition 26, but differs in details.

1. Again, linearity is obvious. Let $v \in \ker \xi_s \setminus \{0\}$ be arbitrary and use the right-hand decomposition in Lemma 15, $v = \sum_{j=0}^{m-1} e_s^j v_j$ with $v_j \in \ker \partial_s^{\text{op}}$. Inserting this into $\xi_s(v) = 0$ yields

$$v_{m-1} = \sum_{j=0}^{m-2} \mu_j e_s^{j+1} v_j$$

for some $\mu_j \in \mathbb{K} \setminus \{0\}$. Due to the linear independence in the decomposition in Lemma 15, we conclude $v_j = 0$ for all j and thus $v = 0$.

2. Straightforward, see Proposition 26.

3. Let $v \in \ker \partial_t$ be arbitrary. Due to Proposition 11 $(\partial_s^{\text{op}})^{m-1}$ and ∂_t commute, so $(\partial_s^{\text{op}})^{m-1}(v) \in \ker \partial_t$. The other summand vanishes due to $\partial_t(e_s v) = e_s \partial_t(v) = 0$.

4. Analog to Proposition 26. \square

We will mainly use the shifts ξ_t in the following, due to the special form of Grañas Freeness Theorem we are using.

Lemma 28 **28**

Let X' be a non-empty subset of X . Let G be a quotient of $\text{Env } X$ with $s^m = e$ for all $s \in X \setminus X'$, such that $X \setminus X'$ still generates G . Then $\bigcap_{t \in X'} \ker \partial_t$ is G -balanced.

Proof By Proposition 24, $K := \bigcap_{t \in X'} \ker \partial_t$ has a G -homogeneous basis. For any $s \in X \setminus X'$ and $g \in G$, ξ_s will map $K \cap \mathfrak{B}(g)$ to $K \cap \mathfrak{B}(sg)$ by Proposition 27, so $\dim(K \cap \mathfrak{B}(g)) = \dim(K \cap \mathfrak{B}(sg))$. The Lemma now follows from the assumption that G is generated by $X \setminus X'$. \square

Theorem 29 **29**

Assume $n \mid m$. Let X' be a non-empty proper subrack of X , and assume that $X \setminus X'$ still generates $\text{Inn } X$. Set \mathfrak{B}' to be the sub-Nichols algebra generated by X' . Then holds

$$\dim \mathfrak{B} = \# \text{Inn } X \cdot \dim \mathfrak{B}' \cdot \dim \left(\mathfrak{B}(e) \cap \bigcap_{t \in X'} \ker \partial_t \right). \quad (10)$$

Proof The proof follows directly from Grañas Freeness Theorem by applying Lemma 28 to Lemma 23 (note that $g_t^m = e$ holds for the inner group if and only if $n \mid m$). \square

Assume X is indecomposable and consists of at least three elements (there is no indecomposable quandle of two elements). Choose $X' = \{t\} \subsetneq X$, thus $\mathfrak{B}' \cong \mathbb{K}[t]/t^n$. Due to irreducibility, there must exist $r, s \in X \setminus X'$ with $r \triangleright s = t$, so $g_t = g_r g_s g_r^{-1}$ and $\text{Inn } X$ is generated by $X \setminus X'$. Applying Theorem 29, we find $\# \text{Inn } X \cdot n \mid \dim \mathfrak{B}$.

Example 30 **30**

From the examples of subsection 2.3, four Nichols algebras fulfill $n \mid m$:

\mathfrak{B}	m	$\dim \mathfrak{B}$	$\#G$	\mathfrak{B}'	$\#G \cdot \dim \mathfrak{B}'$	$\frac{\dim \mathfrak{B}}{\#G \cdot \dim \mathfrak{B}'}$
$\mathfrak{B}(Q_{3,1}, -1)$	2	12	6	$\mathbb{K}[t]/t^2$	12	1
$\mathfrak{B}(Q_{4,1}, \chi_4)$	3	5,184	12	$\mathbb{K}[t]/t^3$	36	144
$\mathfrak{B}(Q_{6,1}, -1)$	2	576	24	$\mathfrak{B}(Q_{3,1}, -1)$	288	2
$\mathfrak{B}(Q_{10,1}, -1)$	2	8,294,400	120	$\mathfrak{B}(Q_{6,1}, -1)$	69,120	120

where we use $G = \text{Inn } X$.

It is still unclear, whether $\mathfrak{B}(Q_{15,7}, -1)$ (the Nichols algebra of the transpositions in the symmetric group \mathfrak{S}_6 with constant cocycle -1) is finite-dimensional or not. If it is, its dimension must be divisible by

$$\#\mathfrak{S}_6 \cdot \dim \mathfrak{B}(Q_{10,1}, -1) = 720 \cdot 8,294,400 = 5,971,968,000.$$

Taking a look at the quotients $\frac{\dim \mathfrak{B}}{\#G \cdot \dim \mathfrak{B}'}$ in the above table, one might guess that $\dim \mathfrak{B}(Q_{15,7}, -1)$ will probably be about at least another factor 720 larger, and thus divisible by 4,299,816,960,000.

5 General Case

We remember from Lemma 21 that $\text{Inn } X$ is a quotient of the shift group Φ if $n \mid m$. Moreover, this induces a G -grading on $\mathbb{K}\Phi$ (which is balanced if $\mathbb{K}\Phi$ is finite-dimensional), which in turn induces a balanced G -grading on \mathfrak{B} . One might ask, how to generalize this idea to the case $n \nmid m$.

Let $\text{ev}_1 : \mathbb{K}\Phi \rightarrow \mathfrak{B}$ be the evaluation at $1 \in \mathfrak{B}$. From Lemma 18 we know that ev_1 is a surjective linear map. ev_1 is neither an algebra homomorphism, nor is its kernel an ideal of $\mathbb{K}\Phi$; still, there is an identification of $\mathbb{K}\Phi / \ker \text{ev}_1$ and \mathfrak{B} as linear spaces. Assume there is a surjective homomorphism $\pi : \Phi \rightarrow G$ to some finite quotient G of $\text{Env } X$. Define $\Phi_g := \pi^{-1}(g)$ and $U_g := \mathbb{K}\Phi_g \subseteq \mathbb{K}\Phi$, such that $\mathbb{K}\Phi = \bigoplus_{g \in G} U_g$. Choose a system of representatives $\phi_g \in U_g \setminus \{0\}$ and define the translations $\tau_g : \mathbb{K}\Phi \rightarrow \mathbb{K}\Phi$, $\phi \mapsto \phi_g \phi$. Each τ_g is a linear isomorphism of $\mathbb{K}\Phi$ and $\tau_g(U_h) = U_{gh}$ for each $g, h \in G$. Now assume $\phi \in \ker \text{ev}_1$. Then $\tau_g(\phi)(1) = \phi_g(\phi(1)) = 0$, hence $\tau_g(\ker \text{ev}_1) = \ker \text{ev}_1$. We may therefore define linear maps $\tilde{\tau}_g : \mathbb{K}\Phi / \ker \text{ev}_1 \rightarrow \mathbb{K}\Phi / \ker \text{ev}_1$ with $\tilde{\tau}_g(U_h / \ker \text{ev}_1) = U_{gh} / \ker \text{ev}_1$. Obviously, we have $\mathbb{K}\Phi / \ker \text{ev}_1 = \sum_{g \in G} U_g / \ker \text{ev}_1$. Assume this sum is direct. Then the isomorphisms $\tilde{\tau}_g$ show that this grading is balanced, and hence $\#G$ divides $\dim \mathfrak{B}$. So there currently are two open questions to transfer the results of Subsection 3.1 to the general case:

1. Is G a quotient of Φ ?
2. Is the sum $\sum_{g \in G} U_g / \ker \text{ev}_1$ direct?

We will now concentrate on $G = C_k$, which is a quotient of $\text{Env } X$ by $t \mapsto [1]_k \in C_k$ for all $t \in X$.

5.1 Factors in the Hilbert Series

Each Nichols Algebra \mathfrak{B} is \mathbb{Z} -graded. Taking quotients, we find C_k -gradings of \mathfrak{B} for each $k > 1$.

Lemma 31 31

\mathfrak{B} is C_m -balanced. In particular, we have for each $[k]_m \in C_m$

$$\sum_{\substack{j \in \mathbb{N}_0, \\ j \equiv k \pmod{m}}} \dim \mathfrak{B}(j) = \frac{1}{m} \dim \mathfrak{B}.$$

Proof Let $t \in X$ be arbitrary. We may define ϕ_t as in Proposition 17 (or, equivalently, any of the shifts of Definition 25) and find that ϕ_t maps $\mathfrak{B}(j)$ to $\mathfrak{B}(j+1) \oplus \mathfrak{B}(j-m+1)$. Hence, ϕ_t is a linear isomorphism between $\mathfrak{B}([j]_m)$ and $\mathfrak{B}([j+1]_m)$. \square

Clearly, if $j \mid k$ and \mathfrak{B} is C_k -balanced, then \mathfrak{B} is C_j -balanced as well. The following table shows for some Nichols algebras \mathfrak{B} those $k > 1$ such that \mathfrak{B} is C_k -balanced.

\mathfrak{B}	n	m	$\dim \mathfrak{B}$	C_k -balanced for ...
$\mathbb{K}[t]/t^m$	1	m	m	$k \mid m$
$\mathfrak{B}(Q_{3,1}, -1)$	2	2	12	$k = 2, 3$
$\mathfrak{B}(Q_{3,1}, E_3)^{(2)}$	2	3	432	$k = 2, 3, 4, 6$
$\mathfrak{B}(Q_{4,1}, -1)^{(2)}$	3	2	36	$k = 2, 3$
$\mathfrak{B}(Q_{4,1}, -1)^{(\neq 2)}$	3	2	72	$k = 2, 3, 6$
$\mathfrak{B}(Q_{4,1}, \chi_4)$	3	3	5,184	$k = 2, 3, 4, 6$
$\mathfrak{B}(Q_{5,*}, -1)$	4	2	1,280	$k = 2, 4, 5$
$\mathfrak{B}(Q_{6,*}, *)$	2/2/4	2	576	$k = 2, 3, 4$
$\mathfrak{B}(Q_{7,*}, -1)$	6	2	326,592	$k = 2, 3, 6, 7$
$\mathfrak{B}(Q_{10,1}, *)$	2	2	8,294,400	$k = 2, 3, 4, 5, 6$

Lemma 32 32

A finite-dimensional Nichols algebra \mathfrak{B} is C_k -balanced (as a quotient of its \mathbb{Z} -grading) if and only if $(k)_t := \sum_{j=0}^{k-1} t^j$ is a divisor of the Hilbert series $\mathcal{H}_{\mathfrak{B}}(t)$ of \mathfrak{B} , such that the quotient polynomial has integer coefficients only.

Proof To simplify notation, if p is a polynomial, set p_j to be the coefficient of t^j in $p(t)$ (or zero if $j < 0$) and $b_j := (\mathcal{H}_{\mathfrak{B}})_j = \dim \mathfrak{B}(j)$ for any $j \in \mathbb{N}_0$.

“ \Rightarrow ”: Set $p_j := 0$ for each $j < 0$ and inductively define $p_j := b_j - \sum_{i=1}^{k-1} p_{j-i}$, hence $b_j - b_{j-1} = p_j - p_{j-k}$. Let $d \in \mathbb{N}_0$ be such that $d \cdot k$ is larger than the top degree of \mathfrak{B} . Then summation of the previous equation yields for each $0 \leq l \leq k-1$ a telescoping sum

$$\begin{aligned} \sum_{0 \leq j \leq d} b_{jk+l} - \sum_{0 \leq j \leq d} b_{jk+l-1} &= \sum_{0 \leq j \leq d} (p_{jk+l} - p_{jk+l-k}) \\ &= -p_{l-k} + p_{dk+l}. \end{aligned}$$

\mathfrak{B} is C_k -balanced by assumption, so the two sums on the left hand side must sum to the same value (namely $\frac{1}{k} \dim \mathfrak{B}$). p_{l-k} is zero by definition ($l-k < 0$), hence p_{dk+l} is zero as well. This shows that p actually is a polynomial, and by definition its coefficients are integers. From $b_j = \sum_{i=0}^{k-1} p_{j-i}$, we also see $\mathcal{H}_{\mathfrak{B}}(t) = (k)_t \cdot p(t)$.

“ \Leftarrow ”: Let $\mathcal{H}_{\mathfrak{B}}(t) = (k)_t \cdot p(t)$ for some polynomial p with integer coefficients. We have $b_j = \sum_{i=0}^{k-1} p_{j-i}$ and therefore for each $[l]_k \in C_k$

$$\dim \mathfrak{B}([l]_k) = \sum_{\substack{j \in \mathbb{N}_0, \\ j \equiv l \pmod{k}}} \dim \mathfrak{B}(j) = \sum_{\substack{j \in \mathbb{N}_0, \\ j \equiv l \pmod{k}}} \sum_{i=0}^{k-1} p_{j-i} = \sum_{j \in \mathbb{N}_0} p_j,$$

which does not depend on $[l]_k$, so \mathfrak{B} is C_k -balanced. \square

From Lemmas 31 and 32 follows that $(m)_t$ is a divisor of the Hilbert series of \mathfrak{B} . This result is well-known and can be seen directly from any of the Freeness Theorems applied to a trivial subrack.

Theorem 33 **33**

Let \mathfrak{B} be a finite-dimensional Nichols algebra over a rack X and a 2-cocycle of order m . Let X' be a non-empty proper subrack of X and \mathfrak{B}' its corresponding Nichols sub-algebra of \mathfrak{B} . Then the Hilbert series $\mathcal{H}_{\mathfrak{B}}(t)$ is divisible by $(m)_t \cdot \mathcal{H}_{\mathfrak{B}'}(t)$.

Proof We use the notation of Lemmas 31 and 32. Let $t \in X$ be arbitrary. We have seen in the proof of Lemma 31, that ϕ_t is a linear isomorphism between $\mathfrak{B}([j]_m)$ and $\mathfrak{B}([j+1]_m)$ for all j ; so is ξ_t from Proposition 27. Applying the same techniques of the proof of Theorem 29 to the quotient C_m -grading yields the proposition. \square

Corollary 34 **34**

Let \mathfrak{B} be a finite-dimensional Nichols algebra over an indecomposable rack X with $\#X \geq 3$ and a 2-cocycle of order m . Then its Hilbert series $\mathcal{H}_{\mathfrak{B}}(t)$ is divisible by $(m)_t^2$.

Example 35 **35**

We know that there is a sequence of embeddings of quandles

$$\{t\} \hookrightarrow Q_{3,1} \hookrightarrow Q_{6,1} \hookrightarrow Q_{10,1}$$

associated to the Nichols-algebra-embeddings

$$\mathbb{K}[t]/t^2 \hookrightarrow \mathfrak{B}(Q_{3,1}, -1) \hookrightarrow \mathfrak{B}(Q_{6,1}, -1) \hookrightarrow \mathfrak{B}(Q_{10,1}, -1).$$

In addition to this, one easily sees that $Q_{6,1} \setminus Q_{3,1}$ still generates $\text{Inn } Q_{6,1}$ and $Q_{10,1} \setminus Q_{6,1}$ still generates $\text{Inn } Q_{10,1}$. Applying Theorem 33 three times now shows that $(2)_t^4$ is a factor of $\mathcal{H}_{\mathfrak{B}(Q_{10,1}, -1)}(t)$. Following Example 30, we conclude that $(2)_t^5$ must be a factor of $\mathcal{H}_{\mathfrak{B}(Q_{15,7}, -1)}(t)$, if this Nichols algebra is finite dimensional.

5.2 The 72-dimensional Nichols Algebra

We now concentrate on one example with $n \nmid m$, the 72-dimensional Nichols algebra $\mathfrak{B}(Q_{4,1}, -1)^{(\neq 2)}$ first introduced in [10].

Let $X = (\{1, 2, 3, 4\}, \triangleright)$ be the quandle with operation

\triangleright	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Choose $\dim V_0 = 1$ and as cocycle choose the constant cocycle $\chi = -1$ over any field \mathbb{K} of characteristic $\neq 2$. The resulting Nichols Algebra \mathfrak{B} has dimension 72 ([10]), a possible basis is given by the following products, written in syntax notation:

$$[e_1] [e_2 [e_1]] [e_3 e_2 e_1] [e_3 [e_2]] [e_4]$$

(each argument in square brackets is optional). Its relations are generated by the relations

$$\begin{aligned} 0 &= e_t^2 \quad \forall t \in X \\ 0 &= e_r e_s + e_s e_t + e_t e_r \\ &\quad \forall (r, s, t) \in \{(4, 3, 2), (4, 2, 1), (4, 1, 3), (3, 1, 2)\} \\ 0 &= (e_3 e_2 e_1)^2 + (e_2 e_1 e_3)^2 + (e_1 e_3 e_2)^2 \end{aligned}$$

The inner group of X is isomorphic to the alternating group A_4 . With respect to this grading, \mathfrak{B} is not balanced:

$g \in A_4$	$()$	$(1, 3)(2, 4)$	$(1, 4)(2, 3)$	$(1, 2)(3, 4)$
$\dim \mathfrak{B}(g)$	12	4	4	4
$g \in A_4$	$(1, 2, 3)$	$(1, 3, 4)$	$(1, 4, 2)$	$(2, 3, 4)$
$\dim \mathfrak{B}(g)$	6	6	6	6
$g \in A_4$	$(1, 3, 2)$	$(1, 4, 3)$	$(1, 2, 4)$	$(2, 4, 3)$
$\dim \mathfrak{B}(g)$	6	6	6	6

(Elements in cycle notation; calculations have been performed with Rig, see [12].) As one sees, the dimension is preserved by conjugation; this is due to the operation of $\text{Env } X$ on \mathfrak{B} , which conjugates the grading.

The grading of \mathfrak{B} with respect to the enveloping group $\text{Env } X$ of X must be unbalanced, because $\text{Env } X$ is infinite. Indeed, the two elements $g_3 g_2 g_1$ and $g_2 g_3 g_2 g_1 g_3 g_4 \in \text{Env } X$ fulfill $\dim \mathfrak{B}(g) = 5$, eight elements have $\dim \mathfrak{B}(g) = 3$, another eight elements 2, 22 elements have $\dim \mathfrak{B}(g) = 1$ (including the identity element) and the remaining elements 0. One would therefore ask, whether there is a quotient G of $\text{Env } X$, such that \mathfrak{B} is G -balanced and G is large enough to have $\text{Inn } X$ as a quotient itself. However:

Proposition 36 **36**

There is no quotient G of $\text{Env } X$, such that $\text{Inn } X$ is a quotient of G and \mathfrak{B} is G -balanced.

Proof Taking a quotient cannot lower the dimensions of the grade. For $g = g_3 g_2 g_1 \in \text{Env } X$ we therefore find, that the dimension of the grade of the image of g under the canonical projection $\text{Env } X \rightarrow G$ (and hence for each element in G) must be at least 5. By hypothesis, $\text{Inn } X$ is a quotient of G and therefore fulfills $\dim \mathfrak{B}(g) \geq 5$ for each $g \in \text{Inn } X$: Contradiction. \square

Proposition 37 **37**

The shift group Φ of $\mathfrak{B}(Q_{4,1}, -1)^{(\neq 2)}$ is infinite in characteristic 0.

Proof The endomorphism $\phi_1 \phi_2$ has a Jacobi normal form with eight blocks of each of the three types

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \rho & 1 & 0 \\ 0 & \rho & 1 \\ 0 & 0 & \rho \end{pmatrix} \quad \begin{pmatrix} \bar{\rho} & 1 & 0 \\ 0 & \bar{\rho} & 1 \\ 0 & 0 & \bar{\rho} \end{pmatrix}$$

where ρ and $\bar{\rho}$ are different third roots of unity. From this decomposition one sees that $\phi_1 \phi_2$ has infinite order if \mathbb{K} is of characteristic 0, and thus $\langle \phi_1, \phi_2 \rangle \cong C_2 * C_2$. \square

Acknowledgements. The author wants to thank István Heckenberger and Leandro Vendramin for their valuable hints and corrections.

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